



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE GEOMETRIC PROPERTIES OF QUARTIC CURVES POSSESSING FOURFOLD SYMMETRY WITH RESPECT TO A POINT *

BY R. D. CARMICHAEL

IN a previous paper † I have considered the problem of classifying, by the form of their equations, plane algebraic curves possessing fourfold symmetry with respect to a point. The center of symmetry being taken as the origin, it is there shown that all such curves are of two classes: and for quartic curves it readily follows that the equations of the two classes in their most general forms are respectively:—

$$(A) \quad 2a_0(x^4 + y^4) + 4a_1xy(x^2 - y^2) + 4a_2x^2y^2 + a_3(x^2 + y^2) + a_4 = 0,$$

$$(B) \quad r_0(x^4 - y^4) + 2r_1(x^2 + y^2) + r_2(x^2 - y^2) + 2r_3xy = 0.$$

In this paper attention will first be given to the classification of the irreducible forms of these curves according to the number of their singularities of each kind. For this purpose we shall require certain of Plücker's equations. By $n, m, \delta, \tau, \rho, \iota$, we shall, as usual represent respectively the order, class, number of double points, number of double tangents, number of cusps, and number of points of inflection of the curve. The Plücker equations are then the following:—

$$(1) \quad m = n(n - 1) - (2\delta + 3\rho),$$

$$(2) \quad n = m(m - 1) - (2\tau + 3\iota),$$

$$(3) \quad \iota = 3n(n - 2) - (6\delta + 8\rho),$$

$$(4) \quad \rho = 3m(m - 2) - (6\tau + 8\iota).$$

From the fact that these curves possess fourfold symmetry with respect to the origin it follows that singularities not at the origin can enter only by fours.

* Read before the American Mathematical Society, October 26, 1907.

† ANNALS OF MATHEMATICS, ser. 2, vol. 9, pp. 53-56; January 1908.

Moreover, even at the origin if one cusp exists there must be three others corresponding to it; for, if not, the curve would not coincide with its former position after one-fourth revolution about the origin. Hence the number of cusps is a multiple of four, or zero. But since $m + 2\delta$ is positive and n is four it follows from (1) that ρ is less than four. Hence ρ is equal to zero; that is, the curves in consideration are non-cuspidal. Now, a quartic curve has not more than three double points; and therefore these in consideration can have no double point except at the origin.

In group *A*, if $a_4 \neq 0$, the locus does not pass through the origin, and hence the curve has no double point. If $a_4 = 0$ and $a_3 \neq 0$, there is a double point at the origin. If both $a_4 = 0$ and $a_3 = 0$, the equation becomes homogeneous in x, y and hence breaks up into linear factors. The locus is then a group of straight lines and is therefore excluded from further consideration. In group *B* there is always a double point at the origin if r_2 and r_3 are not both zero; and if $r_2 = r_3 = 0$, the equation breaks up into factors and the curve is reducible. If there is no double point, that is, if $\delta = 0$, it follows from equations (1) to (4) that $m = 12, \iota = 24, \tau = 28$. If there is one double point, or $\delta = 1$, we have $m = 10, \iota = 18, \tau = 16$. Tabulating we have:

	n	m	δ	ρ	τ	ι
Group <i>A</i> ; Case I, $a_4 \neq 0$	4	12	0	0	28	24
Group <i>A</i> ; Case II, $a_4 = 0$	4	10	1	0	16	18
Group <i>B</i>	4	10	1	0	16	18

Since δ is in no case greater than one, none of these curves is unicursal. The first one belongs to Salmon's group I and the other two to his group II.*

We shall take up these cases separately and determine for each the general form of the locus and certain other geometric properties in addition to those already given.

Group A, Case I. Making the substitution $x = \rho \cos \theta$ and $y = \rho \sin \theta$, and reducing to more convenient form, we have the following polar equation:

$$(5) \quad [2a_0 + (a_2 - a_0) \sin^2 2\theta + a_1 \sin 4\theta] \rho^4 + a_3 \rho^2 + a_4 = 0.$$

* See Salmon-Fiedler's *Analytische Geometrie der höheren ebenen Kurven*, second ed., p. 278.

The coefficient of ρ^4 may be written in the form

$$2a_0 + \frac{a_2 - a_0}{2} + a_1 \sin 4\theta + \frac{a_0 - a_2}{2} \cos 4\theta,$$

which by the substitution of $\theta - \phi$ for θ becomes

$$2a_0 + \frac{a_2 - a_0}{2} + \sin 4\theta \left(a_1 \cos 4\phi + \frac{a_0 - a_2}{2} \sin 4\phi \right) \\ + \cos 4\theta \left(\frac{a_0 - a_2}{2} \cos 4\phi - a_1 \sin 4\phi \right).$$

Putting

$$\tan 4\phi = \frac{a_0 - a_2}{2a_1}$$

in the above expression and substituting in (5), we may write the result in the form

$$(6) \quad (A_1 + B_1 \sin 4\theta) \rho^4 + C_1 \rho^2 + a_4 = 0,$$

$$\text{where} \quad A_1 = 2a_0 + \frac{a_2 - a_0}{2} \quad \text{and} \quad B_1 = \frac{4a_1^2 + (a_0 - a_2)^2}{2a_1} \cos 4\phi.$$

Since $a_4 \neq 0$ we may divide by it and write the result thus:

$$(A + B \sin 4\theta) \rho^4 + C \rho^2 + 1 = 0,$$

$$\text{where} \quad A = \frac{A_1}{a_4}, \quad B = \frac{B_1}{a_4}, \quad C = \frac{C_1}{a_4}.$$

If $C = 0$ the equation reduces to

$$\rho = \pm \sqrt[4]{\frac{-1}{A + B \sin 4\theta}}.$$

The discussion is entirely analogous to that for the general equation in Case II of Group A below. It is unnecessary to give it in detail.

If $C \neq 0$, we have

$$(7) \quad \rho = \pm \sqrt{\frac{-C \pm \sqrt{C^2 - 4r}}{2r}},$$

where $r = A + B \sin 4\theta$. The two values of ρ corresponding to θ and $\theta + 180^\circ$ are evidently the same, and consequently the first double sign in (7) may be omitted and the resulting equation will give, when θ varies from 0° to 360° , every point of the locus which is obtained when the double sign is kept. Hence we may write

$$(8) \quad \rho = \sqrt{\frac{-C \pm \sqrt{C^2 - 4r}}{2r}}.$$

If $r < 0$, there is one real value of ρ .

If $r = 0 -$, there is a finite real value of ρ when $C < 0$, and an infinite real value of ρ when $C > 0$.

If $r > C^2/4$, there is no real value of ρ .

If $r = C^2/4$, there are two coincident values of ρ when $C < 0$, but no real value when $C > 0$.

If $r > 0$ but $< C^2/4$, there are two real and distinct values of ρ when $C < 0$, but no real value when $C > 0$.

If $r = 0 +$, there is a finite real value and an infinite real value of ρ when $C < 0$, but no real value when $C > 0$.

Now, if B is positive,

$$A - B \leq r \leq A + B.$$

If B is negative it should be replaced by $-B$ in the following discussion. A discussion of the different possible figures in terms of A , B , C and hence in terms of the a 's offers but little difficulty.

1) If $A < -B$, r is always negative and the locus is a simple closed curve.

2) If $A = -B$, r is negative or zero. When $r \neq 0$ there is a single finite real ρ for every θ . When $r = 0 -$ there is a finite value of ρ if $C < 0$, and the locus is a closed curve. If $C > 0$, the curve has infinite branches, of which the asymptotes are given by $r = 0$.

3) If $B > A > -B$, r has both positive and negative as well as zero values. If $C > 0$ we have a single branch in each quadrant, corresponding to a negative r , and approaching as asymptotes the lines $r = 0$. No real points correspond to values of θ for which r is positive.

* That is, approaches zero through negative values.

That the lines $r = 0$ are asymptotes of the curve in this case and in the preceding case may be proved by showing that the distance of a variable point (ρ, θ) of the curve from the line $r = 0$, which is $\rho \sin (\alpha - \theta)$, where α is the value of θ for which $r = 0$, approaches zero as θ approaches α . We have

$$\begin{aligned} A + B \sin 4\theta &= r, & A + B \sin 4\alpha &= 0, \\ r &= B (\sin 4\theta - \sin 4\alpha) = 2B \sin 2(\theta - \alpha) \cos 2(\theta + \alpha). \end{aligned}$$

Then $\rho \sin (\alpha - \theta)$ approaches zero as $[\sin (\alpha - \theta)]^{\frac{1}{2}}$, unless $\cos 2\alpha = 0$ or $B = 0$. If $B = 0$ we have also $A = 0$ and the curve is excluded from our consideration, being of the second degree; if $\cos 2\alpha = 0$, we have $A = 0$ and $\sin 4\alpha = 0$, so that $r = B \sin 4\theta$. It may be seen that this case is no exception, but that here also $\rho \cos (\alpha - \theta)$ approaches zero as $[\sin (\alpha - \theta)]^{\frac{1}{2}}$.

To discuss the form of the curve when C is negative, consider first a value of θ for which r is negative, and let θ change through 90° so that r increases to zero, becomes positive, then decreases to zero, becomes negative, and returns to its original value. There is at first one value of ρ remaining finite as $r = 0 -$; when r increases through zero there is a finite value and an infinite value ($r = 0 +$), so that the finite branch is continued and a second branch has the line $r = 0$ as an asymptote. That $r = 0$ gives an asymptote may be proved as before. If r remains less than $C^2/4$, these branches remain distinct, the infinite branch having a second asymptote when $r = 0 +$, so that the locus is a closed curve and four infinite branches like these of a hyperbola. If, however, r becomes greater than $C^2/4$, the two branches come together and have a common tangent from the origin for $r = C^2/4$, and for values greater than $C^2/4$ no real values of ρ . If $A + B = C^2/4$, so that $C^2/4$ is the maximum value of r , we should apparently have a double point, not at the origin, and so contradictory to our previous result. It may, however, be easily seen that in this case the locus is not irreducible, but consists of two hyperbolas. It is of course not proved that the curve is irreducible in other cases.

4) If $A = B$, r is equal to or greater than zero. If $C > 0$, there is no real locus. If $C < 0$, there are two distinct values of ρ , one becoming infinite with $r = 0$, and the locus is as in the previous case where $A + B < C^2/4$.

5) Finally, if $A > B$, $r > 0$. When $C > 0$, the curve has no real locus. When $C < 0$, we have two finite values of ρ , always distinct unless $A - B \leq C^2/4 \leq A + B$. If $C^2/4$ lies outside of these limits the locus consists of two separate closed curves, each surrounding the origin; if between

$A - B$ and $A + B$, the curve consists of four separate closed curves, each excluding the origin, whose tangents from the origin are given by $r = C^2/4$; if finally $C^2/4$ is equal to $A - B$ or $A + B$, the locus reduces to two hyperbolas.

Group A, Case II. Putting $\alpha_4 = 0$ in equation (6), and dividing by $-C_1$, we may write as the polar equation of the curves to be considered here,

$$(9) \quad (A + B \sin 4\theta) \rho^4 - \rho^2 = 0,$$

where the values of A and B are easily found in terms of the α 's. From (9) follows

$$(10) \quad \rho^2 = 0,$$

$$(11) \quad \rho = \sqrt{\frac{1}{A + B \sin 4\theta}}.$$

Corresponding to the values of ρ given by (10) is simply the origin, an isolated point. In (11) the double sign has been omitted for the same reason as that given in the discussion leading to (8).

We have seen that this locus has 18 points of inflection. Moreover, those not at the origin enter by fours. Hence the isolated point at the origin must also count as at least two points of inflection. If there are more than two points of inflection at the origin there must be six, so as to leave those not at the origin a multiple of four. But six coincident points on a quartic is impossible; and therefore the origin counts as just two points of inflection.

Again setting $r = A + B \sin 4\theta$, it will not be difficult to discuss the locus of (11) in terms of A and B . We have as before

$$A - B \leq r \leq A + B,$$

if B is positive, with the same modification as before if B is negative.

1) If $A \leq -B$, there is no real locus.

2) If $B > A > -B$, r has both positive and negative values. Negative values give no real points. Positive values of r give the locus consisting of four branches, each like that of a hyperbola and having two asymptotes given by $r = 0$.

3) If $A = B$, the locus consists, as in the previous case, of four branches

extending to infinity, each having an asymptote common with the next, so that the asymptotes are along two perpendicular lines.

4) Finally, if $A > B$, we have $r > 0$, and there is then a single closed branch.

Group B. Transforming the equation by the substitution $x = \rho \cos \theta$ and $y = \rho \sin \theta$, and reducing, we have:

$$(r_0 \cos 2\theta + r_1 \sin 2\theta) \rho^4 + (r_2 \cos 2\theta + r_3 \sin 2\theta) \rho^2 = 0.$$

Replacing θ by $\theta - \phi$ and setting

$$\tan 2\phi = \frac{r_0}{r_1},$$

the equation takes the form

$$\rho^4 A_1 \sin 2\theta + \rho^2 (B_1 \sin 2\theta + C_1 \cos 2\theta) = 0,$$

where

$$A_1 = r_0 \sin 2\phi + r_1 \cos 2\phi,$$

$$B_1 = r_2 \sin 2\phi + r_3 \cos 2\phi,$$

$$C_1 = r_2 \cos 2\phi - r_3 \sin 2\phi,$$

If $A_1 = 0$ this equation is of the second degree and need not be considered. If $A_1 \neq 0$ we may replace the equation by

$$\rho^4 \sin 2\theta + \rho^2 (B \sin 2\theta + C \cos 2\theta) = 0,$$

where

$$B = B_1/A_1 \text{ and } C = C_1/A_1.$$

This equation is equivalent to the two

$$\rho^2 = 0,$$

$$\rho = \pm \sqrt{-B - C \cotn 2\theta}.$$

The first of these equations gives only the origin. The second is a circle, real or imaginary, if C is zero. If C is not zero the locus consists of two real branches, each having an inflexion at the origin and approaching both ends of one of the coordinate axes asymptotically.